MATHEMATICAL CONTROL THEORY SOLUTION OF AN INTERACTIVE ACCOUNTING FLOWS MODEL*

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ABSTRACT

In this paper we have applied the mathematical control theory to the accounting network flows, where the flow rates are constrained by linear inequalities. The optimal control policy is of the "generalized bang-bang" variety which is obtained by solving at each instant in time a linear programming problem whose objective function parameters are determined by the "switching function" which is derived from the Hamiltonian function. The interpretation of the adjoint variables of the control problem and the dual evaluators of the linear programming problem demonstrates an interesting interaction of the cross section phase of the problem, which is characterized by linear programming, and the dynamic phase of the problem, which is characterized by control theory.

1. INTRODUCTION

This paper is a sequel to our previous paper [1] which dealt with a wheat trading model in which only one product is to be bought and sold subject to maximum purchase and sales limits, when the price variation over time is known with certainty.

In this paper, we shall extend the applications of mathematical control theory to multiple product cases where more than one product is to be bought (produced) and sold under price certainty, subject to various kinds of linear constraints on purchases (production) and sales. We will use notation similar to that used in [1].

Our results are as follows: The adjoint variables have interpretations similar to those in the previous paper [1]. The optimal control policy is of the "generalized bang-bang" variety which is obtained by solving at each instant in time a linear programming problem whose objective function parameters are determined by the "switching functions" of the control problem. The optimal policy then "bangs" from one extreme point of this linear programming problem to another as the switching functions change over time. A simple example is worked out to show how the optimal policy changes as prices change over time.

2. THE CONTINUOUS JOINT TRADING MODEL

We shall describe this model in terms of a trading situation, but it can easily be reinterpreted in terms of a model in which there is the joint production and sale of two different products.

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Let us consider a firm that engages in trading two kinds of commodities, wheat and rice. The firm's assets are of only three kinds, cash, wheat, and rice, whose balances at time t are represented by $c(t)$, $w(t)$, and $v(t)$, respectively. These stocks are measured as follows: $c(t)$ in dollars, and $w(t)$ and $v(t)$ in bushels or in some other physical units. The price of wheat and the price of rice from time 0 to time T are assumed to be known in advance with certainty and are given by functions $p_1(t)$ and $p_2(t)$, respectively. It is also assumed that for each commodity there is only one price at any point in time which is applicable both for buying and selling the commodity. We shall let $P_1 = p_1(T)$ and $P_2 = p_2(T)$ be the terminal prices of wheat and rice, respectively. We assume that the initial assets $C_0 = c(0)$, $W_0 = w(0)$, and $V_0 = v(0)$ are known constants.

The firm's objective is to buy and sell wheat and rice during the time period 0 to T so as to maximize the total value of its assets at time T. Hence, it always values its current inventories with the terminal prices in making trading decisions. Thus the total value of the assets of the firm at time t, denoted by $x(t)$, is given by

$$x(t) = c(t) + P_1 w(t) + P_2 v(t). \tag{1}$$

We denote the rates of buying wheat and rice at time t by $m_{1p}(t)$ and $m_{2p}(t)$, respectively, and the rates of selling wheat and rice at time t by $m_{1s}(t)$ and $m_{2s}(t)$, respectively. They are all expressed in physical units of wheat and rice. We shall impose the following constraints on these variables which are assumed to reflect the buying and selling environments of the firm's operations.

$$\begin{aligned} A_p m_p &\leq b_p \\ A_s m_s &\leq b_s \\ m_p, m_s &\geq 0, \end{aligned} \tag{2}$$

where the matrices A_p $(k \times 2)$, A_s $(n \times 2)$, b_p $(k \times 1)$, and b_s $(n \times 1)$ are given constant matrices, and m_p is the column vector with components m_{1p} and m_{2p}, and m_s is the column vector with components m_{1s} and m_{2s}.

Sometimes there are additional constraints that connect buying and selling activities of the firm of the form,

$$D_p m_p + D_s m_s \leq b_d, \tag{3}$$

where the matrices D_p $(h \times 2)$, D_s $(h \times 2)$, and b_d $(h \times 1)$ are given constant matrices, and D_p and D_s each containing at least one nonzero constant in each row.

The only assumption that we shall make concerning the convex set defined by the constraints (2) and (3) is that there is *at least one* feasible vector satisfying them. Thus the constraint set may be unbounded. As we shall see, this may lead to a maximizing linear programming problem with unbounded solution without affecting our main results. Of course, in any reasonable application the solution to such a maximizing problem will turn out to be bounded since unbounded solutions cannot be realized in practice.

Many other kinds of constraints can be imagined. For instance, constraints (such as those involving liquidity or cash flow) over time, or constraints involving both state and control variables. However, we shall restrict this paper to the consideration of problems that involve only the constraints in the form of (2) and (3) and that there is at least one feasible vector satisfying them.

As long as the constraints (2) and (3) are satisfied, the firm is permitted to be either long or short in any or all of the three asset types, cash, wheat, and rice. A long position is indicated by a positive stock of an asset and a short position by a negative stock.

There are four factors that change the cash balance. One is the interest received on cash (or paid on a loan) balance at a given constant rate R per unit period, which is to be accrued continuously and is to be received (or paid) in cash. That is, the interest on a cash or loan balance is compounded continuously at interest rate R. The second factor is the inventory holding costs $h_1(w)$ for wheat and $h_2(v)$ for rice that must be paid in cash on inventories of sizes $w(t)$ and $v(t)$ at time t. The functions h_1 and h_2 are piecewise differentiable functions of w and v, respectively. They are increasing functions of w and v for their nonnegative values and decreasing functions for the nonpositive values. Thus we interpret the inventory holding cost to be the cost of being short in wheat or rice when $w < 0$ or $v < 0$, which increases as $|w|$ or $|v|$ increases. The third factor changing the cash balance is the cost of purchasing wheat and/or rice, $p_1(t)m_{1p}(t) + P_2(t)m_{2p}(t)$ at time t. The fourth factor is the revenue from sales of wheat and rice, $p_1(t)m_{1s}(t) + p_2(t)m_{2s}(t)$. Thus, the differential equation that expresses the manner in which the stock of cash changes is

$$\dot{c}(t) = Rc(t) - h_1(w) - h_2(v) - p_1(t)m_{1p}(t) - p_2(t)m_{2p}(t) + p_1(t)m_{1s}(t) + p_2(t)m_{2s}(t). \tag{4}$$

There are only two factors that affect the stock of wheat and rice, namely the purchase and sales quantities. Thus, the differential equations for w and v are

$$\dot{w}(t) = m_{1p}(t) - m_{1s}(t) \tag{5}$$

and

$$\dot{v}(t) = m_{2p}(t) - m_{2s}(t). \tag{6}$$

In order to simplify notation, we shall drop (t) from all expressions with the understanding that lower case letters (except b and e) represent functions of time unless other independent variables are specified (e.g., $h(w)$), and that all upper case letters represent constants.

To put the problem into standard Mayer form, we differentiate (1) obtaining:

$$\begin{aligned}\dot{x} = \dot{c} + P_1\dot{w} + p_2\dot{v} = Rc - h_1(w) - h_2(v) - p_1m_{1p} - p_2m_{2p} \\ + p_1m_{1s} + p_2m_{2s} + P_1m_{1p} - P_1m_{1s} + P_2m_{2p} - P_2m_{2s}.\end{aligned} \tag{7}$$

The latter expression was obtained by using (4)–(6).

The firm's objective function is now expressed by

$$\text{Max } x(T) = c(T) + P_1w(T) + P_2v(T) \tag{8}$$

subject to the differential equations (4)–(7) and the constraints (2) and (3).

In order to solve the problem, we introduce adjoint variables $\Psi(t)$ for (4), $\phi(t)$ for (5), $\rho(t)$ for (6), and the constant function -1 for (7). Then the Hamiltonian function is given by

$$\begin{aligned}\mathscr{H}(t) &= \Psi(Rc - h_1(w) - h_2(v) - p_1m_{1p} - p_2m_{2p} + p_1m_{1s} + p_2m_{2s}) + \phi(m_{1p} - m_{1s}) \\ &\quad + \rho(m_{2p} - m_{2s}) - (Rc - h_1(w) - h_2(v) - p_1m_{1p} - p_2m_{2p} + p_1m_{1s} + p_2m_{2s} \\ &\quad + P_1m_{1p} - P_1m_{1s} + P_2m_{2p} - P_2m_{2s}) \\ &= (\Psi - 1)(Rc - h_1(w) - h_2(v)) - \mathscr{A}_1(m_{1p} - m_{1s}) - \mathscr{A}_2(m_{2p} - m_{2s}),\end{aligned} \tag{9}$$

where $\mathscr{A}_1$ and $\mathscr{A}_2$ are the *switching functions* defined as

$$\mathscr{A}_1 = P_1 + p_1(\Psi - 1) - \phi \tag{10}$$

and

$$\mathscr{A}_2 = P_2 + p_2(\Psi - 1) - \rho. \tag{11}$$

The determination of Ψ, ϕ, and their interpretations together with those of $\mathscr{H}$, $\mathscr{A}_1$ and $\mathscr{A}_2$ are given in Section 4. They are entirely analogous to the corresponding functions in Reference [1].

The maximum principle [2], provides as a necessary condition for an optimal policy that the control variables m should be chosen so as to minimize $\mathscr{H}$, which implies the maximization of the criterion function

$$z = \mathscr{A}_1(m_{1p} - m_{1s}) + \mathscr{A}_2(m_{2p} - m_{2s}) = \mathscr{A}(m_p - m_s), \tag{12}$$

subject to the control variable constraints (2) and (3). In other words, at each point of time, the optimal policy involves the solution of an ordinary linear programming problem. The coefficients of the objective function in (12) are the switching functions $\mathscr{A}_1$ and $\mathscr{A}_2$. Since these are functions of time, it follows that what is involved is a parametric solution of this linear programming problem. We summarize our results in the following theorem.

THEOREM: If the constraint set defined by (2) and (3) is nonempty, then the optimal control variables m are determined by solving at each time t the linear programming problem:

$$\operatorname{Max} \mathscr{A}(m_p - m_s).$$

Subject to

$$\begin{aligned} A_pm_p &\leq b_p, \\ A_sm_s &\leq b_s \\ D_pm_p + D_sm_s &\leq b_d \\ m_p, m_s &\geq 0, \end{aligned} \tag{13}$$

where $\mathscr{A} = (\mathscr{A}_1, \mathscr{A}_2)$ is given by (10) and (11).

PROOF: The calculations above show that necessary conditions for optimal control are that the control variables should solve at each point in time the linear programming problem (13). We must still show that these necessary conditions are sufficient. The sufficiency follows immediately, however, by observing that the constraints of the linear programming problem can be assumed (by well-known results) to be nondegenerate so that the solution to the linear programming problem is unique, except for the finite number of values of $\mathcal{S}_1$ and $\mathcal{S}_2$. The example in the next section illustrates the result.

It should be remarked that, if the solution to (13) is infinite, as may well happen for certain values of the switching function in the case that the constraint set is unbounded, our conclusion is still valid even though this gives an infinite value to the Hamiltonian. It may also happen that both m_p and m_s become infinite while the quantity $m_p - m_s$ remains finite, so that the Hamiltonian remains finite. This case also causes no difficulty. Obviously neither of these possibilities will arise in a problem derived from an actual application since solutions involving infinite rates of buying cannot realistically be carried out over any finite length of time.

There are two special cases worthy of mention and further detail. The first case is that of *separable constraints* which occurs when constraints of the form (3) are absent. Here the linear programming problem (13) breaks up into two smaller problems and hence it may be considerably easier to solve. The second case is that of *coordinated control* which occurs whenever $m_p^{tr} \cdot m_s = 0$. In other words, a coordinated control problem is one in which there is never a reason to simultaneously buy and sell at a given time. Again, in realistic applications this is most likely to be the case because of the normal workings of the market.

3. AN EXAMPLE

In this section, we shall analyze the multiple product control problem discussed above by a simple numerical example.

Let us suppose that the rates of inventory holding cost $h_1(w)$ and $h_2(w)$ are specifically given as follows:

$$h_1(w) = 0.20\,|w| \tag{14}$$

$$h_2(v) = 0.10\,|v| \tag{15}$$

The interest rate is 0.01. Both the rate of inventory holding cost and the interest rate are expressed as monthly rates. We shall consider $T = 20$ months for the planning period, and assume the terminal prices of wheat and rice to be \$15 and \$20, respectively. In this analysis, we shall also assume (for simplicity) that the signs of w and v are always positive, i.e., the firm never runs into a short position for the inventories. If the problem involves changes in the sign of w or v, the problem can be solved by the techniques described in our previous paper [1].

Using the result obtained in the previous paper, we can write the switching function $\mathcal{S}_1$ and $\mathcal{S}_2$ as

$$\mathcal{S}_1 = e^{R(T-t)}(u_1 - p_1) \tag{16}$$

and

$$\mathcal{S}_2 = e^{R(T-t)}(u_2 - p_2), \tag{17}$$

where

$$u_1 \equiv \left[\frac{h_1'(w)}{R} + P_1\right] e^{-R(T-t)} - \frac{h_1'(w)}{R} \tag{18}$$

and

$$u_2 \equiv \left[\frac{h_2'(v)}{R} + P_2\right] e^{-R(T-t)} - \frac{h_2'(v)}{R}. \tag{19}$$

Substituting $T=20$, $h_1'(w)=0.20$, $h_2'(v)=0.10$, $P_1=15$, $P_2=20$, and $R=0.01$, we have

$$u_1 = 35e^{-0.01(20-t)} - 20 \tag{20}$$

and

$$u_2 = 30e^{-0.01(20-t)} - 10. \tag{21}$$

In order to simplify the subsequent computations, let us assume the following values of p_1 and p_2.

$$\begin{cases} p_1 = 35e^{-0.01(20-t)} - t - 20 & \text{for } \ 0 \leqslant t \leqslant 4 \\ p_1 = 35e^{-0.01(20-t)} + t - 28 & \text{for } \ 4 \leqslant t \leqslant 14 \\ p_1 = 35e^{-0.01(20-t)} - t & \text{for } 14 \leqslant t \leqslant 20 \end{cases} \tag{22}$$

$$\begin{cases} p_2 = 30e^{-0.01(20-t)} + t - 16 & \text{for } \ 0 \leqslant t \leqslant 12 \\ p_2 = 30e^{-0.01(20-t)} - t + 8 & \text{for } 12 \leqslant t \leqslant 19 \\ p_2 = 30e^{-0.01(20-t)} + t - 30 & \text{for } 19 \leqslant t \leqslant 20. \end{cases} \tag{23}$$

The numerical values at the end of each month based on (22) and (23) are given for p_1 and p_2 in Table 1.

TABLE 1

End of month	p_1	p_2	End of month	p_1	p_2	End of month	p_1	p_2
0	8.66	8.56	7	9.73	17.34	14	18.96	22.25
1	7.95	9.81	8	11.04	18.60	15	18.29	21.53
2	7.24	11.06	9	12.35	19.87	16	17.63	20.82
3	6.53	12.31	10	13.67	21.14	17	16.97	20.11
4	5.82	13.56	11	14.99	22.41	18	16.31	19.40
5	7.12	14.82	12	16.31	23.69	19	15.65	18.70
6	8.42	16.08	13	17.63	22.97	20	15.00	20.00

The figures in Table 1 are only samples, since p_1 and p_2 must be specified for all values of t and not limited to its integer values in the continuous wheat and rice trading model. It is important here to emphasize that the above functions for prices are selected only for the sake of simple illustrations and that the solution algorithm described in this paper can be applied to arbitrary price functions.

From (22), (23) and (20), (21), we obtain the values of $u_1 - p_1$, and $u_2 - p_2$ as follows:

$$\begin{cases} u_1 - p_1 = t & \text{for } 0 \leqslant t \leqslant 4 \\ u_1 - p_1 = 8 - t & \text{for } 4 \leqslant t \leqslant 14 \\ u_1 - p_1 = t - 20 & \text{for } 14 \leqslant t \leqslant 20 \end{cases} \tag{24}$$

$$\begin{cases} u_2 - p_2 = -t + 6 & \text{for } 0 \leqslant t \leqslant 12 \\ u_2 - p_2 = t - 18 & \text{for } 12 \leqslant t \leqslant 19 \\ u_2 - p_2 = -t + 20 & \text{for } 19 \leqslant t \leqslant 20. \end{cases} \tag{25}$$

We now consider the constraints on the control variables. Suppose that the purchases constraints and the sales constraints are set up as follows:

$$\begin{cases} m_{1p} + m_{2p} \leqslant 6. \\ m_{1p} + 2m_{2p} \leqslant 10 \\ 3m_{1p} + m_{2p} \leqslant 12 \\ m_{1p}, \quad m_{2p} \geqslant 0 \end{cases} \tag{26}$$

$$\begin{cases} m_{1s} + m_{2s} \leqslant 5 \\ 3m_{1s} + 2m_{2s} \leqslant 12 \\ m_{1s}, \quad m_{2s} \geqslant 0. \end{cases} \tag{27}$$

These constraints are shown in Figure 1(a) and 1(b). Note that since (26) and (27) are the only constraints, this problem is a separable control problem.

The solutions to the linear programming problems discussed earlier must take on one of the extreme points shown in the above figure or on a point of a convex combination of two adjacent extreme points. Which extreme point is selected depends upon the ratio of $\frac{\Delta_1}{\Delta_2}$. For example, if $\frac{\Delta_2}{\Delta_1} = 1$, then (2, 4) and (3, 3) are equally preferred since $2\Delta_1 + 4\Delta_2 = 3\Delta_1 + 3\Delta_2$. But if $\frac{\Delta_2}{\Delta_1} > 1$ then (2, 4) becomes better. As $\frac{\Delta_2}{\Delta_1}$ becomes larger and larger eventually (0, 5) is preferred to (2, 4) which occurs when $\frac{\Delta_2}{\Delta_1} > 2$. Similarly as $\frac{\Delta_2}{\Delta_1} < 1$, (3, 3) is preferred to (2, 4) and as $\frac{\Delta_2}{\Delta_1} < \frac{1}{3}$, (4, 0) becomes preferable to (3, 3). If Δ_1 and Δ_2 are both negative, the sales constraints become effective. If $\frac{\Delta_2}{\Delta_1} > 1$, (0, 5) is better than (2, 3), but as $\frac{\Delta_2}{\Delta_1} < 1$ the latter is preferred until $\frac{\Delta_2}{\Delta_1} < \frac{2}{3}$ at which time (4, 0) is preferred to (2, 3).

These may be summarized in Figure 2. The numbers in brackets show the optimum values of the control variables, the first number for m_{1p} or m_{1s} and the second number for m_{2p} or m_{2s}. If the number is positive, then that number is the value of m_{1p} or m_{2p}, depending upon the element, and m_{1s} or m_{2s} is set equal to zero. If the number is negative, the negative of that number is the value of m_{1s} or m_{2s}, depending upon the element, and m_{1p} or m_{2p} is set equal to zero.

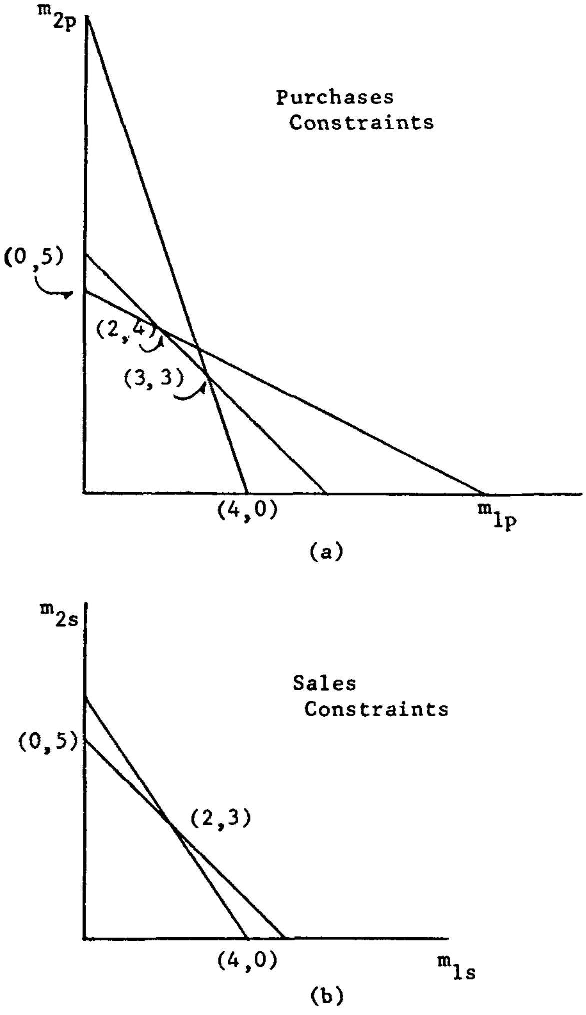

FIGURE 1

Now we relate this analysis with the values of $u_i - p_i$ $(i=1, 2)$ which obtained earlier. Since only the ratio of Δ_1 and Δ_2 is crucial, and since $\frac{\Delta_2}{\Delta_1} = \frac{u_2 - p_2}{u_1 - p_1}$, we may use $u_1 - p_1$ and $u_2 - p_2$ in place of Δ_1 and Δ_2. Then with the equations in (24) and (25) together with Figure 2, we obtain Figure 3. The broken line indicates the change in $u_1 - p_1$ and $u_2 - p_2$ as t changes. The numbers represent the month.

This figure completely specifies the optimum strategy. For example, during Month 2, $\frac{u_2 - p_2}{u_1 - p_1} > 2$, hence we should purchase only rice at the rate of 5 units per month. However, $\frac{u_2 - p_2}{u_1 - p_1}$ decreases over time and becomes equal to 1 at the end of Month 2, hence the strategy for Month 3 is to purchase 2 units of wheat and 4 units of rice. Similarly, in Month 13 we should sell only rice at the rate of 5 units a month and Month 14 we should sell 2 units of wheat and 3 units of rice per month.

During the last 2 months sales of wheat at the rate of 4 units a month and purchases of rice at the rate of 5 units a month are required. These are summarized in Table 2.

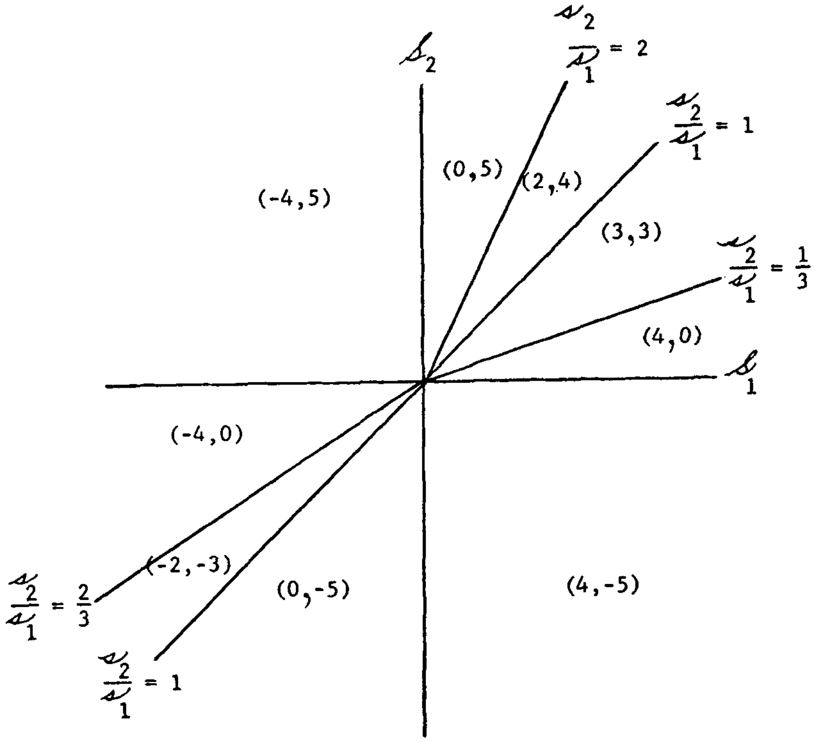
$\mathscr{S}_2$
$\frac{\mathscr{S}_2}{\mathscr{S}_1} = 2$
$\frac{\mathscr{S}_2}{\mathscr{S}_1} = 1$
(0,5)
(2,4)
(-4,5)
(3,3)
$\frac{\mathscr{S}_2}{\mathscr{S}_1} = \frac{1}{3}$
(4,0)
$\mathscr{S}_1$
(-4,0)
(-2,-3)
$\frac{\mathscr{S}_2}{\mathscr{S}_1} = \frac{2}{3}$
(0,-5)
(4,-5)
$\frac{\mathscr{S}_2}{\mathscr{S}_1} = 1$

FIGURE 2

TABLE 2

Optimum monthly purchase or sales rate		
Months	Wheat	Rice
1, 2	No trade	Buy 5 units
3	Buy 2 units	Buy 4 units
4, 5	Buy 3 units	Buy 3 units
6	Buy 4 units	No trade
7, 8	Buy 4 units	Sell 5 units
9–13	No trade	Sell 5 units
14	Sell 2 units	Sell 3 units
15–18	Sell 4 units	No trade
19, 20	Sell 4 units	Buy 5 units

In the above example, we designed the prices, for the sake of simple illustrations, in such a way that changes in the control variables occur only at the end of each month. Generally, this is not true, and mid-period corrections in the control variables are often required in continuous control problems.

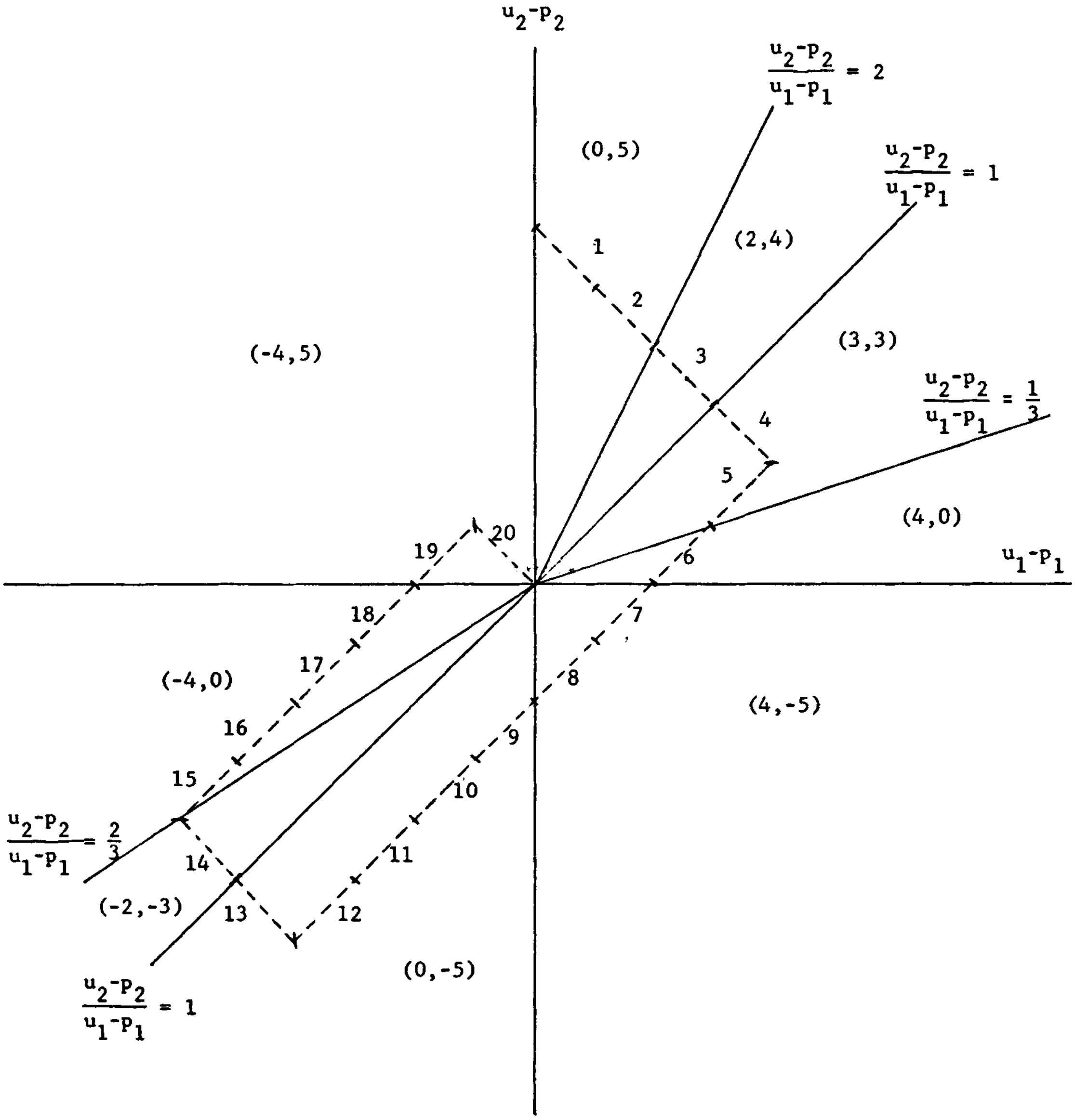
u_2-p_2
$\frac{u_2-p_2}{u_1-p_1} = 2$
$\frac{u_2-p_2}{u_1-p_1} = 1$
$\frac{u_2-p_2}{u_1-p_1} = \frac{1}{3}$
$\frac{u_2-p_2}{u_1-p_1} = \frac{2}{3}$
$\frac{u_2-p_2}{u_1-p_1} = 1$
u_1-p_1
(0,5)
(2,4)
(3,3)
(4,0)
(-4,5)
(-4,0)
(-2,-3)
(0,-5)
(4,-5)
1
2
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FIGURE 3

4. INTERPRETATIONS OF THE OPTIMAL POLICY

Since the derivations and interpretations of the formulas for the adjoint functions ϕ and ρ, the switching functions $\mathscr{A}_1$ and $\mathscr{A}_2$, and the Hamiltonian function $\mathscr{H}$ are virtually identical to those given in our previous paper [1], we shall only briefly discuss them here.

To obtain the differential equation for Ψ we calculate

$$\dot{\Psi} = -\frac{\partial \mathscr{H}}{\partial c} = -(\Psi - 1)R.$$

The solution to this equation with boundary condition $\Psi(T) = 0$ is

$$\Psi(t) = 1 - e^{R(T-t)}. \tag{28}$$

The interpretation of $\Psi(t)$ is that it is *the net increase in value of a dollar invested from time* t *to* T *at interest rate* R.

The differential equations for ϕ and ρ are obtained as follows:

(w)
$$\dot{\phi} = -\frac{\partial \mathscr{H}}{\partial w} = (\Psi - 1)h_1'(w)$$

$$\dot{\rho} = -\frac{\partial \mathscr{H}}{\partial v} = (\Psi - 1)h_2'(v).$$

Substituting (28) into these equations and integrating, we obtain

$$\phi = \int_t^T h_1'(w)e^{R(T-\tau)}d\tau \tag{29}$$

and

$$\rho = \int_t^T h_2'(v)e^{R(T-\tau)}d\tau. \tag{30}$$

As in [1] the interpretation of (29) or (30) is *the total time-adjusted unit cost of storing a small increment of wheat or rice from time* t *to* T.

Substituting these results into the switching functions (10) and (11) we have

$$\mathscr{S}_1 = P_1 - p_1e^{R(T-t)} - \phi \tag{31}$$

and

$$\mathscr{S}_2 = P_2 - p_2e^{R(T-t)} - \rho. \tag{32}$$

The interpretation of $\mathscr{S}_1$ is *the future* (*time* T) *price of wheat*, P_1, *less the future value of* $p_1e^{R(T-t)}$ *of paying now* (*time* t) *for wheat at price* p_1, *less the total storage cost* ϕ *from time* t *to* T *of the additional wheat bought now.* The interpretation of $\mathscr{S}_2$ is similar.

Substituting the results into (9) gives the following form for the Hamiltonian:

$$-\mathscr{H} = [Rc - h_1(w) - h_2(v)] - \mathscr{S}(m_p - m_s). \tag{33}$$

From this we can interpret the negative of the Hamiltonian as follows: *the function* $-\mathscr{H}$ *is the future* (*time* T) *value of the present* (*time* t) *cash flow and consists of two parts: the first part is the cash flow due to previous* (*before time* t) *decisions, and the second part is the cash flow due to the present* (*time* t) *decisions.*

It is obvious that we can extend these results to much more general problems with many more control variables, and more complicated constraints. We shall carry out such a generalization in a later paper in this series.

One final interpretation is now possible concerning the dual variables to the maximizing linear programming problem (13). It is well known that the dual to a maximizing problem is a minimizing problem and that one of these problems has a finite solution if and only if the other does. The interpretation of the dual solution depends upon the specific units of the various quantities involved. As an example, assume that A_p, A_s, D_p, and D_s in (13) are pure numbers. Since the units of the switching functions Δ_1 and Δ_2 and measured in dollar per unit and m_p and m_s are in units per time period we conclude that b_p, b_s, and b_d have units per time period also. Hence the dual variables to the linear programming problem (13), call them $z_1, \ldots z_m$, have units of dollar per time period. This means that $z_1(t)$ for example, measures the *rate of contribution to the final asset position if we relax the righthand side of the first constraint by one unit now (time* t). This is a rather remarkable interpretation for it evaluates an action now in terms of future values.

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